A NOTE ON THE JENSEN INEQUALITY FOR SELF-ADJOINT OPERATORS. II.

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Abstract. This is a continuation of our previous paper. We consider a certain order-like relation for positive operators on a Hilbert space. This relation is defined by using the Jensen inequality with respect to the square-root function. We show that this relation is antisymmetric if the operators are invertible.

1. Introduction

This is a continuation of our previous paper [7]. Let $f(t)$ be a continuous, increasing concave function on the half line $[0, \infty)$ and let A and B be bounded self-adjoint operators on a Hilbert space $\mathfrak{H}$ with an inner product $\langle \cdot, \cdot \rangle$. In the previous paper, we consider the following problem. If A and B satisfy $\langle f(A)\xi, \xi\rangle \leq f(\langle B\xi, \xi\rangle)$ and $\langle f(B)\xi, \xi\rangle \leq f(\langle A\xi, \xi\rangle)$ for any unit vector $\xi \in \mathfrak{H}$, can we conclude $A = B$? This problem was suggested by Professor Bourin [4]. In [7] we solved this problem affirmatively in the finite-dimensional case. We also dealt with some related problem in the infinite-dimensional case, but we could not get a complete answer. In this paper we consider the case $f(t) = \sqrt{t}$ and we solve this problem affirmatively under the assumption that two positive operators A and B are both invertible.

For two positive operators A and B, we introduce the new relation $A \trianglelefteq B$ defined by $\langle A^{\frac{1}{2}}\xi, \xi\rangle \leq \langle B\xi, \xi\rangle^{\frac{1}{2}}$ for any unit vector $\xi \in \mathfrak{H}$. Using this notation, we can restate the above problem as follows. If A and B satisfy $A \trianglelefteq B$ and $B \trianglelefteq A$, can we conclude $A = B$? We will show that this is true when A and B are both invertible. Here we remark that the usual order $A \leq B$ implies $A \trianglelefteq B$ thanks to the Jensen inequality. However the relation $\trianglelefteq$ is not an order relation. Indeed we will construct positive matrices A, B and C such that both $A \trianglelefteq B$ and $B \trianglelefteq C$ hold while $A \trianglelefteq C$ does not hold.

The author would like to express his hearty gratitude to Professor Tsuyoshi Ando for valuable comments.

2000 *Mathematics Subject Classification.* 47A63.

Key words and phrases. operator inequality, Jensen inequality.

2. Main Result

Throughout this paper we assume that the readers are familiar with basic notations and results on operator theory. We refer the readers to Conway's book [5].

We denote by $\mathfrak{H}$ a (finite or infinite dimensional) complex Hilbert space and by $B(\mathfrak{H})$ all bounded linear operators on it. The operator norm of $A \in B(\mathfrak{H})$ is denoted by $||A||$. The inner product and the norm for two vectors $\xi, \eta \in \mathfrak{H}$ are denoted by $\langle \xi, \eta \rangle$ and $||\xi||$ respectively. We denote the defining function for an interval $[a, b)$ by $\chi_{[a,b)}(t)$. We define the absolute value for a bounded linear operator X by $|X| = (X^*X)^{\frac{1}{2}}$.

If two positive operators $A, B \in B(\mathfrak{H})$ satisfy

$$\langle A^{\frac{1}{2}}\xi, \xi\rangle \leq \langle B\xi, \xi\rangle^{\frac{1}{2}}$$

for any unit vector $\xi \in \mathfrak{H}$, we write

$$A \trianglelefteq B.$$

The usual order $A \leq B$ implies that $A \trianglelefteq B$. This is a consequence of the famous Jensen inequality as follows.

$$\langle A^{\frac{1}{2}}\xi, \xi\rangle \leq \langle A\xi, \xi\rangle^{\frac{1}{2}} \leq \langle B\xi, \xi\rangle^{\frac{1}{2}}.$$

Here we remark that the relation $\trianglelefteq$ is not an order relation. Indeed there exit positive matrices A, B and C such that both $A \trianglelefteq B$ and $B \trianglelefteq C$ hold while $A \trianglelefteq C$ does not hold. See Example 2.1.

The following is the main result of this paper.

Theorem 2.1. *Let $A, B \in B(\mathfrak{H})$ be two positive operators such that A is invertible. If they satisfy $A \trianglelefteq B$ and $B \trianglelefteq A$, then we have $A = B$.*

Here we remark that it is hard to remove the assumption of invertibility. See Example 2.1.

Proposition 2.2 (Ando [2])**.** *For two positive operators $A, B \in B(\mathfrak{H})$, the following conditions are equivalent.*

(i) $A^2 \trianglelefteq B^2$.
(ii) $A \leq \frac{1}{2t}B^2 + \frac{t}{2}$ *for any positive number* t.
(iii) *There exists a contraction C satisfying* $CB + BC^* = 2A$

Proof. The equivalence (i)$\Leftrightarrow$(ii) is shown in [1]. (See also [7] Lemma 3.2.)

Suppose that there exists a contraction C satisfying $CB + BC^* = 2A$. Since

$$0 \leq (CB - t)^*(CB - t) = BC^*CB + t^2 - t(CB + BC^*),$$

we see that

$$2tA = t(CB + BC^*) \leq BC^*CB + t^2 \leq B^2 + t^2.$$

Therefore the implication (iii)⇒(ii) holds.

Finally we will show (ii)⇒(iii). We remark that the inequality

$$B^2 + t^2 - 2tA \geq 0$$

holds for any real number t. Thus by the operator-valued Fejer-Riesz theorem ([8]Theorem 3.3) there exist two bounded linear operators X and Y such that

$$B^2 + t^2 - 2tA = (X - tY)^*(X - tY) = X^*X + t^2Y^*Y - t(X^*Y + Y^*X).$$

Therefore we have $B = |X|$, $|Y| = 1$ and $2A = X^*Y + Y^*X$. Here we remark that Y is a contraction because $|Y| = 1$. Take a polar decomposition $X = U|X| = UB$ where U is a partial isometry. Then we get

$$2A = B(U^*Y) + (Y^*U)B.$$

Since U^*Y is a contraction, we are done.

□

Lemma 2.3. *Let c and ϵ be positive numbers such that $\epsilon < c$. Then*

$$2t\lambda - t^2 > 0 \quad \text{and} \quad \frac{\lambda^2}{2t} + \frac{t}{2} - (2t\lambda - t^2)^{\frac{1}{2}} \geq 0$$

for any $c + \epsilon \leq t, \lambda \leq 2c$. Further there exists a positive number d satisfying

$$\frac{\lambda^2}{2t} + \frac{t}{2} - (2t\lambda - t^2)^{\frac{1}{2}} \leq \frac{d}{2}(t - \lambda)^2 \tag{9}$$

for any $c + \epsilon \leq t, \lambda \leq 2c$.

Proof. The proof is same as that of [7] Lemma 3.4.

Since $c + \epsilon \leq t, \lambda \leq 2c$, we have

$$2t\lambda - t^2 = t(2\lambda - t) \geq (c + \epsilon)\{2(c + \epsilon) - 2c\} = 2(c + \epsilon)\epsilon > 0.$$

Next by the arithmetic-geometric mean inequality we have $\frac{\lambda^2}{2t} + \frac{t}{2} \geq \lambda$ and obviously $\lambda^2 \geq 2t\lambda - t^2$, so that $\lambda \geq (2t\lambda - t^2)^{\frac{1}{2}}$.

Now we set

$$k(t, \lambda) = \frac{d}{2}(t - \lambda)^2 - \frac{\lambda^2}{2t} - \frac{t}{2} + (2t\lambda - t^2)^{\frac{1}{2}}.$$

Then we compute

$$\frac{\partial}{\partial t}k(t, \lambda) = d(t - \lambda) + \frac{\lambda^2}{2t^2} - \frac{1}{2} + \frac{\lambda - t}{(2t\lambda - t^2)^{\frac{1}{2}}}$$

and

$$\frac{\partial^2}{\partial t^2}k(t, \lambda) = d - \frac{\lambda^2}{t^3} + \frac{-(2t\lambda - t^2)^{\frac{1}{2}} - (\lambda - t)^2(2t\lambda - t^2)^{-\frac{1}{2}}}{2t\lambda - t^2}.$$

Since $c+\epsilon \le t \le 2c$ and $c+\epsilon \le \lambda \le 2c$, we see that $2t\lambda - t^2 = t(2\lambda - t) \ge (c+\epsilon)\{2(c+\epsilon) - 2c\} = 2(c+\epsilon)\epsilon > 0$. Thus the two-variable function

$$-\frac{\lambda^2}{t^3} + \frac{-(2t\lambda - t^2)^{\frac{1}{2}} - (\lambda - t)^2(2t\lambda - t^2)^{-\frac{1}{2}}}{2t\lambda - t^2}$$

is bounded below on the intervals $c+\epsilon \le t \le 2c$ and $c+\epsilon \le \lambda \le 2c$. Therefore we can find a positive constant d such that $\frac{\partial^2}{\partial t^2} k(t,\lambda) > 0$ on the intervals $c+\epsilon \le t \le 2c$ and $c+\epsilon \le \lambda \le 2c$. Then $k(t,\lambda)$ is convex with respect to t. Since $\frac{\partial}{\partial t} k(t,\lambda)|_{t=\lambda} = 0$, $k(t,\lambda)$ in t is decreasing for $c+\epsilon \le t \le \lambda$ and increasing for $\lambda \le t \le c$ so that $k(t,\lambda) \ge k(\lambda,\lambda) = 0$. Thus we are done. □

Lemma 2.4. *Let $A, B \in B(\mathfrak{H})$ be positive invertible operators such that $c+\epsilon \le A \le 2c$ for some positive numbers $\epsilon < c$. If they satisfy*

$$(2tA - t^2)^{\frac{1}{2}} \le B \le \frac{A^2}{2t} + \frac{t}{2}$$

for any positive number t on the interval $c+\epsilon \le t \le 2c$, then we have $A = B$.

Proof. The proof is essentially same as that of [1, 6, 7].

First we will show that there exists a positive constant d satisfying

$$||PBP - (PB^{-1}P)^{-1}|| \le d||tP - AP||^2 \tag{1}$$

for any $c+\epsilon \le t \le 2c$ and any spectral projection P of A, where we use $(PB^{-1}P)^{-1}$ to denote the inverse of $PB^{-1}P$ on $P\mathfrak{H}$. In the following we use commutativity of A and P without any particular mention.

By assumption we have two inequalities

$$(2tA - t^2)^{\frac{1}{2}} \le B \le \frac{A^2}{2t} + \frac{t}{2} \tag{2}$$

and

$$2t(A^2 + t^2)^{-1} \le B^{-1} \le (2tA - t^2)^{-\frac{1}{2}}. \tag{3}$$

Here we remark that $(2tA - t^2)^{-\frac{1}{2}}$ is a bounded operator because $2tA - t^2 = t(2A - t)$ and $2A \ge 2(c+\epsilon) > 2c \ge t > 0$. On the other hand we have

$$(2tA - t^2)^{\frac{1}{2}} \le A \le \frac{A^2}{2t} + \frac{t}{2}. \tag{4}$$

By the inequalities (2) and (4), we see that

$$\pm(AP - PBP) \le \frac{(AP)^2}{2t} + \frac{t}{2}P - (2tAP - t^2P)^{\frac{1}{2}}$$

and hence

$$||AP - PBP|| \le \left|\left|\frac{(AP)^2}{2t} + \frac{t}{2}P - (2tAP - t^2P)^{\frac{1}{2}}\right|\right|. \tag{5}$$

By the inequality (3) we have

$$2t(A^2+t^2)^{-1}P \le PB^{-1}P \le (2tA-t^2)^{-\frac{1}{2}}P$$

and hence

$$(2tAP-t^2P)^{\frac{1}{2}} \le (PB^{-1}P)^{-1} \le \frac{(AP)^2}{2t}+\frac{t}{2}P. \tag{6}$$

By the inequalities (4) and (6) we have

$$\pm(AP-(PB^{-1}P)^{-1}) \le \frac{(AP)^2}{2t}+\frac{t}{2}P-(2tAP-t^2P)^{\frac{1}{2}}$$

and hence

$$||AP-(PB^{-1}P)^{-1}|| \le \left|\left|\frac{(AP)^2}{2t}+\frac{t}{2}P-(2tAP-t^2P)^{\frac{1}{2}}\right|\right|. \tag{7}$$

By the inequalities (5) and (7) we get

$$||PBP-(PB^{-1}P)^{-1}|| \le 2\left|\left|\frac{(AP)^2}{2t}+\frac{t}{2}P-(2tAP-t^2P)^{\frac{1}{2}}\right|\right|. \tag{8}$$

By the inequality (8) and Lemma 2.3 we have shown the inequality (1).

By the well-known formula known as Schur multiplier, we have

$$(PB^{-1}P)^{-1} = PBP - PBP^{\perp}(P^{\perp}BP^{\perp})^{-1}P^{\perp}BP$$

and hence

$$PBP-(PB^{-1}P)^{-1} = PBP^{\perp}(P^{\perp}BP^{\perp})^{-1}P^{\perp}BP \tag{9}$$

with $P^{\perp}=1-P$. Therefore by inequality (1) and (9) we see that

$$||PBP^{\perp}(P^{\perp}BP^{\perp})^{-1}P^{\perp}BP|| \le d||tP-AP||^2 \tag{10}$$

Then by the inequality (10) we compute

$$\begin{aligned}
||P^{\perp}BP||^2 &= ||(P^{\perp}BP^{\perp})^{1/2}(P^{\perp}BP^{\perp})^{-1/2}P^{\perp}BP||^2 \\
&\le ||B||\cdot||(P^{\perp}BP^{\perp})^{-1/2}P^{\perp}BP||^2 \\
&= ||B||\cdot||PBP^{\perp}(P^{\perp}BP^{\perp})^{-1}P^{\perp}BP|| \\
&\le d||B||\cdot||tP-AP||^2
\end{aligned}$$

and hence

$$||P^{\perp}BP||^2 \le d||B||\cdot||tP-AP||^2. \tag{11}$$

For each integer n, let $P_i \quad (i=1,2,\cdots,n)$ be the spectral projections of A corresponding to the interval $[c+\epsilon+\frac{(i-1)\{2c-(c+\epsilon)\}}{n}, c+\epsilon+\frac{i\{2c-(c+\epsilon)\}}{n}]$. Then we have $\sum_i P_i = 1$ and

$$||t_iP_i - AP_i|| \le \frac{c-\epsilon}{n} \tag{12}$$

where $t_i = c + \epsilon + \frac{(i-1)\{2c-(c+\epsilon)\}}{n}$. By the inequalities (11) and (12) we see that

$$\begin{aligned}
||\sum_{i=1}^{n} P_i^{\perp} B P_i||^2 &= ||\{\sum_{i=1}^{n} P_i^{\perp} B P_i\}\{\sum_{j=1}^{n} P_j B P_j^{\perp}\}|| \\
&= ||\sum_{i=1}^{n} P_i^{\perp} B P_i B P_i^{\perp}|| \\
&\leq \sum_{i=1}^{n} ||P_i^{\perp} B P_i B P_i^{\perp}|| \\
&= \sum_{i=1}^{n} ||P_i^{\perp} B P_i||^2 \\
&\leq \sum_{i=1}^{n} d||B|| \cdot ||t_i P_i - A P_i||^2 \\
&\leq \sum_{i=1}^{n} d||B|| \cdot \frac{(c-\epsilon)^2}{n^2} = d||B|| \cdot \frac{(c-\epsilon)^2}{n}
\end{aligned}$$

and hence

$$||\sum_{i=1}^{n} P_i^{\perp} B P_i||^2 \leq d||B|| \cdot \frac{(c-\epsilon)^2}{n}. \tag{13}$$

Since

$$A - B = \sum_{i=1}^{n} (A P_i - P_i B P_i) + \sum_{i=1}^{n} P_i^{\perp} B P_i,$$

by (13) we see that

$$\begin{aligned}
||A - B|| &\leq ||\sum_{i=1}^{n} (A P_i - P_i B P_i)|| + ||\sum_{i=1}^{n} P_i^{\perp} B P_i|| \\
&\leq \sup_i ||A P_i - P_i B P_i|| + \Big(d||B|| \cdot \frac{(c-\epsilon)^2}{n}\Big)^{\frac{1}{2}}
\end{aligned}$$

On the other hand by (5) and Lemma 2.3 we have

$$||A P_i - P_i B P_i|| \leq \frac{d}{2} ||t P_i - A P_i||^2 \leq \frac{d}{2} \Big(\frac{c-\epsilon}{n}\Big)^2$$

Thus we get

$$||A - B|| \leq \frac{d}{2} \Big(\frac{c-\epsilon}{n}\Big)^2 + \Big(d||B|| \cdot \frac{(c-\epsilon)^2}{n}\Big)^{\frac{1}{2}}$$

By tending $n \to \infty$ we see that $A = B$. $\square$

Lemma 2.5. *Let $A, B \in B(\mathfrak{H})$ be positive operators satisfying $A \trianglelefteq B$. If A is invertible, then B is also invertible.*

Proof. By assumption, there exists a positive number c which satisfies $c \leq A$. Then we have

$$c^{\frac{1}{2}}\langle \xi, \xi\rangle \leq \langle A^{\frac{1}{2}}\xi, \xi\rangle \leq \langle B\xi, \xi\rangle^{\frac{1}{2}}$$

for any unit vector $\xi \in \mathfrak{H}$. Therefore B is invertible. □

Lemma 2.6. *Let A be a positive operator and let C be a contraction. If they satisfy $CA + AC^* = 2A$, then we have $CP = P$ where P is the range projection of A.*

Proof. This is a kind of triangle equality. The proof is implicitly contained in [3]. By assumption we have $(C-1)A = A(1-C^*)$. This means that the operator $(C-1)A$ is skew-selfadjoint. Therefore the spectrum $\sigma((C-1)A)$ is contained in $i\mathbb{R}$. On the other hand we see that $\sigma((C-1)A) \cup \{0\} = \sigma(A^{\frac{1}{2}}(C-1)A^{\frac{1}{2}}) \cup \{0\}$, and by [3] Lemma 2.2 we have $\sigma(A^{\frac{1}{2}}(C-1)A^{\frac{1}{2}}) \cap i\mathbb{R} = \{0\}$. Therefore we conclude that $\sigma((C-1)A) = \{0\}$. Since $(C-1)A$ is skew-selfadjoint, we see that $(C-1)A = 0$. □

Proof of Theorem 2.1. By Lemma 2.5 we may assume that both A and B are invertible. It is enough to show that two relations $A^2 \trianglelefteq B^2$ and $B^2 \trianglelefteq A^2$ ensure that $A = B$ for positive invertible operators A and B.

By Proposition 2.2 we have two inequalities

$$A \leq \frac{B^2}{2t} + \frac{t}{2} \tag{14}$$

and

$$B \leq \frac{A^2}{2t} + \frac{t}{2} \tag{15}$$

for any positive number t. Since A is positive invertible, there exists a positive number c satisfying $A \geq c$. Let ϵ be a positive number with $\epsilon < c$. It follows from (14) and Lemma 2.3

$$0 \leq 2tA - t^2 \leq B^2$$

for any $c + \epsilon \leq t \leq 2c$. Then since the map $X \longmapsto X^{\frac{1}{2}}$ is order-preserving in the cone of positive operators, we have from (15)

$$(2tA - t^2)^{\frac{1}{2}} \leq B \leq \frac{A^2}{2t} + \frac{t}{2}.$$

for any $c + \epsilon \leq t \leq 2c$. Let $P = \chi_{[c+\epsilon, 2c]}(A)$. Then we have

$$(2tAP - t^2P)^{\frac{1}{2}} \leq PBP \leq \frac{(AP)^2}{2t} + \frac{t}{2}P$$

and $(c+\epsilon)P \le AP \le 2cP$. Therefore by Lemma 2.4 we have $AP = PBP$. By Proposition 2.2 there exists a contraction D such that

$$DA + AD^* = 2B$$

and hence

$$PDPA + APD^*P = 2PBP = 2AP.$$

Then by Lemma 2.6 we see that $PDP = P$. Since

$$P = PD^*PDP \le PD^*DP \le P,$$

we have $(1-P)DP = 0$ and hence $DP = PDP + (1-P)DP = P$. By the same argument we see that $PD = P$. Therefore we have

$$2BP = (DA + AD^*)P = DPA + AD^*P = 2AP$$

and hence $BP = PB$. Since ϵ is arbitrary, we have

$$A\chi_{(c,2c]}(A) = B\chi_{(c,2c]}(A) = \chi_{(c,2c]}(A)B.$$

Since the positive invertible operators $A(1-\chi_{(c,2c]}(A))$ and $B(1-\chi_{(c,2c]}(A))$ on $(1-\chi_{(c,2c]}(A))\mathfrak{H}$ satisfy

$$\{A(1-\chi_{(c,2c]}(A))\}^2 \trianglelefteq \{B(1-\chi_{(c,2c]}(A))\}^2$$

and

$$\{B(1-\chi_{(c,2c]}(A))\}^2 \trianglelefteq \{A(1-\chi_{(c,2c]}(A))\}^2,$$

by the same argument we see that

$$A\chi_{(2c,4c]}(A) = B\chi_{(2c,4c]}(A) = \chi_{(2c,4c]}(A)B.$$

Therefore by repeating this argument we have $A = B$.

$\square$

Lemma 2.7. *For any operator X, we have*

$$\mathrm{Re}X \le \frac{1}{2t}|X|^2 + \frac{t}{2}$$

for any positive number t.

Proof. Since

$$0 \le (X-t)^*(X-t) = |X|^2 + t^2 - 2t\mathrm{Re}X,$$

we are done. $\square$

Example 2.1. First we will show that there exist 2×2 positive matrices A, B and C such that both $A^2 \trianglelefteq B^2$ and $B^2 \trianglelefteq C^2$ hold while $A^2 \trianglelefteq C^2$ does not hold.

We set

$$X = \begin{pmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{pmatrix}, \qquad A = \mathrm{Re}X = \begin{pmatrix} \sqrt{2} & \frac{1}{2} \\ \frac{1}{2} & \sqrt{2} \end{pmatrix} \ge 0$$

and

$$B = |X| = \frac{1}{3}\begin{pmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 5 \end{pmatrix}.$$

By Lemma 2.7 and Proposition 2.2 we have $A^2 \trianglelefteq B^2$. Next we set

$$Y = \frac{1}{3}\begin{pmatrix} 4 & 2\sqrt{2} \\ 0 & 5 \end{pmatrix}$$

and $C = |Y|$. Since

$$\mathrm{Re}Y = \frac{1}{3}\begin{pmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 5 \end{pmatrix} = B,$$

we have $B^2 \trianglelefteq C^2$. Suppose that $A^2 \trianglelefteq C^2$. Then by Proposition 2.2 we have

$$A \leq \frac{1}{2t}C^2 + \frac{t}{2}$$

for any positive number t. Let $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then we see that $EAE = \sqrt{2}E$ and $E(\frac{1}{2t}C^2 + \frac{t}{2})E = (\frac{1}{2t} \times \frac{16}{9} + \frac{t}{2})E$. Therefore we have

$$\sqrt{2} \leq \frac{8}{9t} + \frac{t}{2}$$

for any positive number t. This is impossible because the minimal value of the right hand side is $\frac{4}{3}$ while $\frac{4}{3} < \sqrt{2}$

Next we show that $(A+\epsilon)^2 \trianglelefteq (B+\epsilon)^2$ is not valid for any positive number ϵ. If this is the case, we have

$$E(A+\epsilon)E = (\sqrt{2}+\epsilon)E \leq \frac{1}{2t}E(B+\epsilon)^2E + \frac{t}{2}E = \Big(\frac{9\epsilon^2 + 24\epsilon + 18}{18t} + \frac{t}{2}\Big)E$$

for any positive number t. Since the minimal value of the scalar on the right hand side is $\frac{\sqrt{9\epsilon^2+24\epsilon+18}}{3}$, we have

$$(\sqrt{2}+\epsilon)^2 = \epsilon^2 + 2\sqrt{2}\epsilon + 2 \leq \Big(\frac{\sqrt{9\epsilon^2 + 24\epsilon + 18}}{3}\Big)^2 = \epsilon^2 + \frac{8}{3}\epsilon + 2.$$

This is obviously wrong because $2\sqrt{2} > \frac{8}{3}$. This is the reason why we cannot remove the assumption of invertibility. In the proof of the main theorem, the inequality

$$A \leq \frac{1}{2t}B^2 + \frac{t}{2}$$

is crucial. So if A is not invertible, we hope that the inequality

$$A + \epsilon \leq \frac{1}{2t}(B+\epsilon)^2 + \frac{t}{2}$$

holds for any small number ϵ. However this is not true in general.

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